

ORIGIN OF QUANTUM RANDOMNESS IN THE PILOT WAVE QUANTUM MECHANICS

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(May 13, 1997)

We account for the origin of the laws of quantum probabilities in the de Broglie-Bohm (pilot wave) formulation of quantum theory by considering the property of ergodicity likely to characterise the dynamics of microscopic quantum systems.

1. OUTLOOK OF THE THEORY. THE PROBLEM

Pilot wave quantum mechanics, the theory put forward by Bohm,⁽¹⁾ and based on the earlier ideas of de Broglie,⁽²⁾ was aimed at resolving the notorious “measurement problem” inherent in quantum theory. The source of this problem can be seen in the twofold meaning attributed to the wave functions that describe the quantum experiment. Wave functions that we ascribe to microscopic invisible quantum systems (such as elementary particles) are eventually used to represent *probability amplitudes* for the observable events that take place with our measuring devices. Wave functions ascribed to the devices themselves receive a vague definition of representing their *states*, that, in turn, must correspond to observers’ sensible impressions of the devices. In such a twofold definition of the basic quantity lies the

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source of difficulties and paradoxes of various kinds, the famous Schrödinger cat paradox being one of them. One of the major merits of the pilot wave formulation is that it overcomes this situation.

The basic idea of the pilot wave theory is the following. Every closed physical system is described by a deterministic evolution of configuration variables (which Bell⁽³⁾ has called “beables”). These are the same as in the classical physics and are just the spatial coordinates of the elementary particles and the configurations of various fields. The evolution of the configuration variables is guided (piloted, in de Broglie’s terminology) by a quantum wave that obeys the Schrödinger equation. The probabilistic character of this mechanics is merely the consequence of our (essential) ignorance of and inability to control the actual values of particle and field microscopic configuration variables.

Besides nonrelativistic quantum mechanics the pilot wave interpretation has been also applied to the relativistic theory of particles and bosonic (scalar and vector) fields, and was argued to be consistent with the observable special relativity.^(4,5) A straightforward extension to quantum geometrodynamics was made by Holland⁽⁵⁾ and by Horiguchi⁽⁶⁾ and further studied by the author.⁽⁷⁾ A complete pilot wave quantum theory of particles and fields (sketched in Ref. 4, see also Ref. 7) still remains to be developed.

Consider the pilot wave theory in more detail in nonrelativistic quantum mechanics. A set of N nonrelativistic spinless particles are described by their spatial coordinates $\mathbf{x} \equiv (\mathbf{x}_1, \dots, \mathbf{x}_N)$. The wave function ψ of such a system obeys the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2} \sum_n \frac{1}{m_n} \Delta_n \psi + V \psi, \quad (1)$$

where $V = V(\mathbf{x})$ is the particle interaction potential. If one represents the wave function in the polar form as $\psi = R \exp(iS/\hbar)$ then from Eq. (1) it follows that the phase $S(\mathbf{x}, t)$ and the amplitude $R(\mathbf{x}, t)$ satisfy the system

$$\frac{\partial S}{\partial t} + \sum_n \frac{1}{2m_n} (\nabla_n S)^2 + V + Q = 0, \quad (2)$$

$$\frac{\partial R^2}{\partial t} + \sum_n \frac{1}{m_n} \nabla_n (R^2 \nabla_n S) = 0, \quad (3)$$

where

$$Q = - \sum_n \frac{\hbar^2}{2m_n} \frac{\Delta_n R}{R} \quad (4)$$

is the so-called quantum potential. In the pilot wave formulation of quantum mechanics the evolution of the coordinates \mathbf{X} , that correspond to the arguments \mathbf{x} of the wave function, is governed by $\psi(\mathbf{x}, t)$ via the guidance equation

$$m_n \dot{\mathbf{X}}_n = \frac{i\hbar}{2} \left(\frac{\psi \nabla_n \psi^* - \psi^* \nabla_n \psi}{|\psi|^2} \right)_{\mathbf{x}=\mathbf{X}} = \nabla_n S(\mathbf{X}, t). \quad (5)$$

The equation (2) for $S(\mathbf{x}, t)$ is the quantum generalisation of the classical Hamilton-Jacobi equation and differs from the latter only by the presence of the quantum potential $Q(\mathbf{x}, t)$. The guidance equation (5) is expressed in terms of the function $S(\mathbf{x}, t)$ in the same form as in the classical theory it is expressed in terms of the solution to the Hamilton-Jacobi equation. Hence, in the limit in which the quantum potential Q in (2) can be neglected we recover classical evolution. We thus see that new formulation of quantum theory can be regarded as just a “deformation” of the classical dynamics (general discussion of this analogy with the classical case can be found in Ref. 5). Note that in the present interpretation the temporal dynamics of the particle coordinates completely determines the physical state of a system, be it microscopic or macroscopic. The role of the wave function in every physical situation is one and the same, namely, to provide the guidance laws for configuration variables. Thus the description of the physical systems becomes unified, and the abovementioned source of the difficulties, which existed in the twofold character of such a description, is eliminated.

The formalism of quantum dynamics outlined above can be readily applied to the case of a single closed quantum system. In practice, however, we usually deal with what we call quantum ensembles that are collections of many identical systems each piloted in the way described above. If all these systems are piloted by one and the same wave function then an ensemble is called pure. Otherwise it is called mixed. (Note that the systems that form such ensembles need not exist simultaneously: experiments with one and the same arrangement may be carried out repeatedly.) In the pilot wave formulation of quantum

mechanics the measurement process is regarded as just a partial case of the generic evolution guided by a wave function that obeys the Schrödinger equation. The probabilistic character of measurement outcomes is caused by our ignorance of and inability to control the actual (initial) values of particle and field microscopic configuration variables in each system of an ensemble as well as in the measuring apparatus.

Consider general description of an ideal measurement of an observable Λ , with a discrete spectrum Λ_n and the corresponding normalised eigenstates $\xi_n(x_S)$, of a system described by a set of coordinates x_S . Let the measuring apparatus be described by a set of variables x_A and let its initial wave function be $\phi(x_A)$. The initial wave function of the total system is

$$\psi_i(x_S, x_A) = \xi(x_S) \phi(x_A) = \sum_n c_n \xi_n(x_S) \phi(x_A) . \quad (6)$$

Suppose that due to interaction between the system and the measuring apparatus the wave function evolves into

$$\psi_f(x_S, x_A) = \sum_n c_n \xi_n(x_S) \phi_n(x_A) , \quad (7)$$

with nonoverlapping, macroscopically distinct, normalised states ϕ_n of the measuring apparatus. In the course of the measurement process the corresponding configuration variables X_S and X_A evolve in a definite way depending on their initial values, and at the end of the experiment the variables X_A turn out to be in a localisation region of one, and only one, of the states ϕ_n . Then the macroscopic state of the apparatus, hence, the measurement outcome, is uniquely specified in an experiment over a particular system.

If we have an ensemble of measurements described by Eqs. (6) and (7) then the outcomes will be random due to random distribution of the initial variables X_S and X_A . In the pilot wave mechanics, in order that the probabilities p_n of different measurement outcomes coincide with those of the standard (Copenhagen) approach, $p_n = |c_n|^2$, it is necessary to assume that the configuration variables of the systems in a quantum ensemble are distributed in accord with their wave function, so that $p(x) = |\psi(x)|^2$, where x denotes the set of all configuration variables, and $p(x)$ is their distribution function. Such a condition is sometimes

called *quantum equilibrium*^(8,9): it is a consequence of the Schrödinger equation that provided the equality holds initially for a given ensemble, it will hold at all times (so long as the ensemble remains closed). However, in the framework of the theory discussed, one has to explain the origin of such a distribution. This question, which is crucial for the pilot wave interpretational scheme, will be the focus of the discussion in this paper.

2. ATTEMPTS AT A SOLUTION

Bohm himself^(10,4) gave qualitative reasoning with regard to this problem. The (typically) complicated, quasirandom, motions of interacting particles, he argued, would lead to the establishment of quantum equilibrium. If one defines function $f(x)$ by $p(x) = f(x)|\psi(x)|^2$, then it is easy to see that $f(x)$ is conserved along the trajectories, and the conjecture made by Bohm was that due to the complicated character of these trajectories the *coarse-grained* value of $f(x)$ will approach unity, thus $p(x)$ will approach $|\psi(x)|^2$ as *coarse-grained* values, what may be sufficient for all practical purposes. However, he has not succeeded in justifying these insights quantitatively. Perhaps for this reason in the modified pilot wave proposal of Bohm and Vigier⁽¹¹⁾ (see also Ref. 4) an additional external stochastic force was added to the right-hand side of the guidance equations (5) in order to account for the occurrence of quantum equilibrium. A similar theory was also put forward by Nelson.⁽¹²⁾ In this paper we consider only the original “minimal” version of the pilot wave theory as it is expressed in Ref. 1.

Among the recent approaches to the problem of quantum equilibrium that we are aware of, one is due to Valentini⁽⁸⁾ and another is due to Dürr, Goldstein and Zanghì.⁽⁹⁾ While the approach of Valentini can be regarded as an elaboration of Bohm’s argument (see above), that of Dürr *et al.* is based on quite a different idea. To our mind, however, the proofs and demonstrations contained in Ref. 8 and in Ref. 9 do not achieve the goal, for the reasons that follow.

Valentini⁽⁸⁾ made an attempt to justify the Bohm’s conjecture that for a pure ensemble

of closed *complicated* systems the coarse-grained distribution $\bar{p}(x)$ of the configuration variables will approach the coarse-grained value $\overline{|\psi(x)|^2}$ (here overline denotes coarse graining). The corresponding analysis involves the quantity $S = -\int \bar{p} \log \left(\bar{p} / \overline{|\psi|^2} \right) dx$ called “subquantum entropy.” By analogy with the classical statistical mechanics (reasoning based on Boltzmann’s H -theorem) it is suggested by the author⁽⁸⁾ that this quantity will increase in time approaching its maximum value of zero, thereby leading to coarse-grained quantum equilibrium, $\bar{p} = \overline{|\psi|^2}$. Such a suggestion is based solely on the fact (called “subquantum H -theorem” by Valentini⁽⁸⁾) that if the conditions $p = \bar{p}$ and $|\psi|^2 = \overline{|\psi|^2}$ (the conditions of “no fine-grained microstructure,” assumed to hold at the initial moment of time) are valid, then the above-presented coarse-grained “entropy” S acquires its local minimum at that moment of time. This property of S , however, does not seem to be sufficient for the conjecture to be justified, if only because it will hold equally well for systems that will never approach quantum equilibrium (see examples in the final part of this paper).

Demonstration of Dürr *et al.*⁽⁹⁾ is based on the notion of *typicality* which is applied to the domain of all possible initial conditions of a model universe. Specifically, the modulus squared $|\Psi|^2$ of the universal wave function is taken to represent the measure density of typicality in the domain of configuration variables. This measure is singled out on the basis of its equivariance, which means that at any moment of time it is expressed through Ψ in one and the same manner. The authors then show that the set of initial conditions that would conform (to certain precision) with the usual quantum mechanical statistical predictions has measure of typicality close to one. To our mind, equivariance of the *specific subjective* measure introduced, although important property, is not sufficient for regarding this measure as relevant to *objective* distributions encountered in the experiments. Especially as it was noted by the authors themselves (Sec. 7 of Ref. 9) that a different choice of the measure for typicality would result in predicted probability distributions different from the observed ones.

3. ERGODICITY ARGUMENT FOR STATIONARY STATES

In this paper we suggest that arguments of the *ergodic theory* can be used to justify the quantum equilibrium hypothesis. An approach of such a kind has been noted by Valentini (see p. 40 of Ref. 8(b)) but rejected in view of one of the difficulties inherent in it, what will be discussed below.

Let us take the point of view, adopted in the classical statistical mechanics (see, e.g., Landau and Lifshits⁽¹³⁾), that equilibrium ensemble averages of various functions of dynamical variables can be represented by their time averages. The theory developed along this line of reasoning is the ergodic theory (for an introduction to which see Ref. 14). A dynamical system in this theory is regarded as a measure space together with one-parameter (discrete or continuous) group of measure-preserving transformations. A subset in the space of dynamical variables is called *invariant set* if it is invariant (modulo set of measure zero) with respect to all these transformations. A dynamical system is called *ergodic* if for any of its invariant sets the measure either of this set, or of its complement, is zero. As a consequence of the Birkhoff-Khinchin ergodic theorem, the fraction of time spent by an ergodic system in a measurable region Ω of its dynamical variables tends to a value proportional to the invariant measure of this region as time goes to infinity. For example, in the case of the Hamiltonian dynamics such an invariant measure is the surface measure, induced by the Liouville measure, on the constant energy surface in the phase space. Justification of the microcanonical equilibrium distribution then reduces to the proof (which is usually a difficult task) or assumption of ergodicity of a particular system. Note, that the ergodicity property can be formulated in terms of *any* measure equivalent¹ to the invariant measure, in this sense ergodicity does not rely strongly on this latter. On the other hand, for an ergodic system the invariant measure is unique in the corresponding equivalence class.

¹Two measures with common domain are said to be *equivalent* if they have as mutual all the sets of measure zero.

If we take all this to be of equal significance to the pilot wave quantum mechanics, we can relate the distribution of the configuration variables to the measure density $|\Psi|^2$ using the ergodicity property of the corresponding pilot wave dynamic flow. We must consider several problems with this approach. The fact that the measure with density $|\Psi|^2$ is in general time-dependent, hence only equivariant rather than also invariant, calls for an essential modification of the above argument, as compared to the classical case [it is this difficulty that has been noted on p. 40 of Ref. 8(b)]. This problem will be simply avoided if one restricts attention to systems in stationary states. This is what we shall do first. Incidentally, this is just what takes place when one proceeds to the *universal* level (as suggested by Dürr *et al.*⁽⁹⁾) and takes into account *general covariance* of the complete theory that includes gravity. One then finds out that the universal wave function does not depend on time² (which is a well-known fact, see, e.g., our paper⁽⁷⁾ for treatment in the pilot wave formulation) so that the corresponding “measure density” *is* invariant. A subsystem of such a universe can happen to be sufficiently “disentangled” from the rest of the world, at the same time exhibiting ergodic dynamics. Since the total wave function is time-independent, the wave function of such a “disentangled” subsystem will also be stationary, and the following reasoning will apply to this subsystem.

Consider, then, a system in a stationary ergodic state with a square integrable wave function Ψ . The ergodicity guarantees that the average time spent by the system in any region of its configuration variables is proportional to the measure of that region with measure density $|\Psi|^2$, as required. For the preparation process the ergodicity argument proceeds as follows. Let $z = (x, y)$ denote the configuration variables of the total system, where x represents the coordinates of the subsystem of interest, and y the coordinates of the environment. Let the total wave function $\Psi(z)$ have a structure

²In some non-standard proposals, like, e.g., in Ref. 8(b), the universal wave function *does* depend on time and does not respect the Wheeler-De Witt equation of the canonical quantum gravity.

$$\Psi(z) = \psi(x)\phi(y) + \Psi_0(z), \quad (8)$$

in which $\phi(y)$ is nonvanishing in a region Ω of the variables y , which is complementary to the y -support of $\Psi_0(z)$. Then every time the corresponding piloted configuration variable Y gets into the region Ω , the configuration variable X is piloted by the wave function $\psi(x)$. Provided Y is in the region Ω , the probability that X will be in a region ω of the variables x is given by the limit of the corresponding time ratio as

$$\mathbb{P}(X \in \omega | Y \in \Omega) = \lim_{T \rightarrow \infty} \frac{\int_0^T \chi_{\omega \times \Omega}(Z(t)) dt}{\int_0^T \chi_{\mathcal{C} \times \Omega}(Z(t)) dt}, \quad (9)$$

where \mathcal{C} is the whole domain of x , and χ_M denotes the characteristic function of a set M . Since the evolution governed by $\Psi(z)$ is ergodic,³ then, according to the Birkhoff-Khinchin ergodic theorem (see Ref. 14), the limit in Eq. (9) exists for almost every initial value of Z and results in

$$\mathbb{P}(X \in \omega | Y \in \Omega) = \frac{\mu_\Psi(\omega \times \Omega)}{\mu_\Psi(\mathcal{C} \times \Omega)} = \mu_\psi(\omega) \equiv \int_\omega d\mu_\psi, \quad (10)$$

where μ_Ψ and μ_ψ are the measures in the domains, respectively, of z and of x with densities determined by the corresponding normalised wave functions. The characteristics of the region Ω disappear from the result (10), and, if necessary, one can apply a formal limit of infinite-dimensional domain of y . The equality (10) in principle constitutes the justification of the standard quantum probabilities.

Of course, there are states that do not lead to ergodic evolution, like, e.g., a state with a real wave function. One should *assume* that the system of relevance is in a state that is close to ergodic. This is our specification of *complicated* systems, or, rather, complicated states. The conditions under which the quantum evolution is ergodic must be further studied, just as it is the case with the classical ergodic theory of equilibrium. We leave this as a matter of future investigation. Some examples are presented in the last section.

³To avoid misunderstanding, note that the dynamics of X when governed by $\psi(x)$ need not be ergodic.

At this point it is worth mentioning one interesting possibility of generating ergodic motions. Let

$$\dot{X} = v(X) \tag{11}$$

be the guidance equation in the original approach of Bohm (as in Eq. (5) for the nonrelativistic case) with the time-independent generalised velocity $v(X)$. To the right-hand side of Eq. (11) one can always add an arbitrary extra term $v'(X)$, such that

$$\operatorname{div}_X \left(|\psi(X)|^2 v'(X) \right) = 0. \tag{12}$$

Then the new guidance equation

$$\dot{X} = v(X) + v'(X) \tag{13}$$

will still define a measure-preserving flow. Presumably the velocity $v'(X)$ can be chosen so complicated that the flow (13) will be (close to) ergodic, and, at the same time the effect of $v'(x)$ will be unobservable on macroscopic scale. The modification (13) is well in the spirit of the proposals of Refs. 11 and 12, the difference is that in our case the extra velocity term $v'(X)$ is not of stochastic nature.

Perhaps, comments are required concerning the nature of the time parameter t in Eq. (9). This parameter is associated with the time translation symmetry of the quantum dynamics of our closed system. With respect to this time parameter the evolution operator $U_t : \mathcal{C}_z \rightarrow \mathcal{C}_z$ that acts in the configuration space \mathcal{C}_z of our system, forms a one-parameter group: $U_{t+s} = U_t U_s$. The integration measure in Eq. (9) is the Lebesgue measure on the real axis of t . This is the only measure that we must take for granted in the present approach. Mathematically, this measure arises due to the ergodic theorem. Physically, it reflects the fact that experiment with the x system in the state $\psi(x)$ will start at a random moment of time with uniform probability distribution, the only natural probability distribution in the context of stationarity.

One can possibly improve the above argument in several ways. One of them is to consider N identical systems described by the corresponding collections of variables x_1, \dots, x_N . The

total wave function will be a function of these variables, as well as of the environment variables y . Let then $z = (x_1, \dots, x_N, y)$ be the whole set of configuration variables, and let the whole range of z contain N regions $\Theta_1, \dots, \Theta_N$ that may overlap, with the following properties: when $z \in \Theta_n$ then the wave function acquires an approximate form

$$\Psi(z) \approx \psi(x_n) \phi(x_1, \dots, \hat{x}_n, \dots, x_N, y), \quad (14)$$

for the values of x_n that form a set of measure μ_ψ close to unity, so that the n th system is guided by the wave function of the form $\psi(x)$. Then one will be interested in the probability that, provided z is in one of the regions Θ_n , the corresponding variable x_n is in a certain region ω in the configuration space of x . This probability, according to the ergodic theory, will again be given by an appropriate time ratio (which one can easily write down) similar to Eq. (9), what will result in the last expression of Eq. (10). In this case, however, an observer will have N systems at his disposal, and a large region $\Theta_1 \cup \Theta_2 \cup \dots \cup \Theta_N$ of “recurrence,” so that equilibrium time average in the ensemble of N systems will be achieved more rapidly as compared to the case of only one such system. Note that the situation just described is analogous to that of real experiments, in which ensembles are usually constituted of many different identical systems.

Next, if a system of interest, which is described by the coordinates x , is part of a *large* closed ergodic system, then it follows that its equilibrium properties will be revealed only on long timescale, namely, on the recurrence timescale of the whole system with respect to the region Ω described above. Therefore one needs ergodicity to take place on a sufficiently small scale. For instance, it may turn out that the coordinates y of the environment can be partitioned in M different ways into y'_m and y''_m , $m = 1, \dots, M$ with the following additional property. When y''_m is in a certain region Ω''_m the wave function (8) acquires the form

$$\Psi(z) = \psi_m(x, y'_m) \phi_m(y''_m), \quad (15)$$

such that the (x, y'_m) system is piloted by the wave function $\psi_m(x, y'_m)$ that itself leads to an ergodic motion. Now, it might turn out that $(\mathcal{C}'_m \times \Omega''_m) \cap \Omega \neq \emptyset$ (Ω being the region

described after Eq. (8) and \mathcal{C}'_m is the whole domain of y'_m) for all or, at least, for several m . In this case the ergodicity argument will apply, at a time, to one of the M subsystems described by the coordinates (x, y'_m) , with smaller recurrence time. In such a way an hierarchy of ergodic motions might take place, resulting in a sufficiently small equilibrium time for the x system. It seems that the conditions of such a kind are likely to occur in nature.

4. ERGODICITY ARGUMENT FOR A GENERIC CASE

It is clear that from the ergodic point of view it is not so much stationarity that is important, as the property that a system of interest acquires a specified wave function $\psi(x)$ repeatedly. Then whenever it is in the state $\psi(x)$ one can apply time averages to its various dynamical variables, and use ergodicity arguments to explain the origin of quantum equilibrium distribution $p(x) = |\psi(x)|^2$. In this section we briefly discuss this more general case.

Consider a subsystem with configuration variables x in an environment with configuration variables y . We do not assume the total system to be in a stationary state. We, however, suppose that in interaction with the environment the x system preserves its identity, and from time to time acquires a specific wave function $\psi(x)$, what means that the total wave function $\Psi(x, y)$ factorises as $\psi(x)\phi_n(y)$ at moments of time t_n , $n = \dots, -1, 0, 1, 2, \dots$. Then it may occur that the dynamics of the configuration variables Y almost does not depend on that of X , while influencing strongly the behaviour of X . For instance, Y may be semiclassical variables of very massive objects. This property may take place at least on certain timescale (of order $t_n - t_{n-1}$) large as compared to the timescale of motion of X , what will be sufficient for the following argument. As an example, the reader may have in mind a gas of molecules, with $\psi(x)$ describing the internal state (electronic configuration) of all the molecules, and y corresponding to their centre-of-mass coordinates. The x system will be perturbed from time to time by the influence from the dynamics of Y . As before, let \mathcal{C} be the space of configuration variables x . Then a sequence of maps $U_n : \mathcal{C} \rightarrow \mathcal{C}$ will

emerge, such that $U_n(x) = X(t_n, x)$, where $X(t_n, x)$ is the configuration of X at the moment t_n provided its configuration at the moment t_{n-1} is x . Due to manifestly chaotic influence from the dynamics of Y the maps U_n in totality are likely to form an ergodic sequence of transformations. This will lead to establishment of quantum equilibrium for the x system in the sense of time averages, as discussed above. Thus processes that occur on relatively large mass scales can influence the small-scale quantum dynamics leading to ergodicity of the latter.

5. EXAMPLES

The following examples will illustrate our approach. Consider a free quantum particle of mass m on a torus $T^n = S^1 \times \cdots \times S^1$ (n times) with period lengths l_1, \dots, l_n . Stationary states with definite momenta are in a standard correspondence with sequences of integers $\{n_i, i = 1, \dots, n\}$. If the squared lengths $\{l_i^2\}$ are rationally independent, then the pilot wave dynamic flow $x_i(t) = x_i(0) + v_i t \pmod{l_i}$ with $v_i = 2\pi\hbar n_i / ml_i$, $i = 1, \dots, n$ will be ergodic for integers $\{n_i\}$ all different from zero. The invariant measure in these states is just the uniform Lebesgue measure, and from the ergodicity argument it follows that the average time spent by the particle in a region Ω will be proportional to the invariant measure of that region. The situation becomes very simple in the particular case of a circle S^1 . That in a stationary state with nonzero momentum the mean time spent by the particle in any segment is proportional to the integral of $|\psi(x)|^2$ over that segment, is easily verified without recourse to the ergodic theory. Note that in the above example the approach of Valentini⁽⁸⁾ would achieve no goal, since any probability distribution $p(x)$ would be simply translated along the torus retaining its shape. From the viewpoint expressed in Ref. 8 the systems just considered would be regarded as not sufficiently complicated.

As a realistic example, consider molecular collisions in a gas. Before two molecules approach each other their electrons behave independently, piloted by intrinsic wave function $\psi(x)$. During a collision the electronic motions become perturbed by interaction, but after

the molecules fly apart the electrons inside each of them are again piloted by the old wave function $\psi(x)$, if the probability of electronic excitation is low (what we assume). Suppose that in the process of collision the motion of molecular centres of mass are to a sufficiently large extent independent of their electrons' motions. Then the process of collision induces a map $U : \mathcal{C}_1 \times \mathcal{C}_2 \rightarrow \mathcal{C}_1 \times \mathcal{C}_2$, where \mathcal{C}_1 and \mathcal{C}_2 are, respectively, the domains of the electrons' configurations x_1 (of the first molecule) and x_2 (of the second molecule). This map preserves the measure with density $|\psi(x_1)\psi(x_2)|^2$ and the sequence of such maps due to repeatable collisions is likely to be ergodic. Due to this ergodicity an equilibrium distribution of the electrons' configurations in molecules will be established.

It is interesting to note that certain cases appear to be tractable in a rather simple way, and do not require quantum equilibrium for *all* the configuration variables prior to the experiment. For example, consider a simplified model (see, e.g., Bohm⁽¹⁵⁾) of the Stern-Gerlach experiment in which a measurement of the electrons' total spin in z -direction is performed in an atom like silver, with zero orbital angular momentum and total electrons' spin equal to $\hbar/2$. Let q denote the coordinates of the centre of mass of the atom, and x the coordinates of the electrons with respect to the centre of mass. The wave function has one free spinor index, and can be presented as $\sum_{\alpha} c_{\alpha} \Psi_{\alpha}(q, x) |\alpha\rangle$, with $\alpha \in \{\uparrow, \downarrow\}$ describing the z -component of the electrons' spin, and c_{α} being constants such that $\sum_{\alpha} |c_{\alpha}|^2 = 1$. The Hamiltonian that describes the experiment contains the part H_{int} that describes interaction of the system with the magnetic field of the Stern-Gerlach apparatus. Since the electrons' wave function is localised within a tiny region around the nucleus, in solving the Schrödinger equation we can approximate H_{int} by $\sigma_z V_q$, where V_q acts only on the centre of mass coordinates q , and σ_z is the Pauli matrix that describes the electrons' spin, so that the total Hamiltonian is

$$H = H_x + H_q + \sigma_z V_q, \quad (16)$$

where H_x and H_q are the corresponding free Hamiltonians for x and q . Let the initial wave function be

$$\Psi(q, x) \sum_{\alpha} c_{\alpha} |\alpha\rangle = \phi(q) \psi(x) \sum_{\alpha} c_{\alpha} |\alpha\rangle, \quad (17)$$

where $\phi(q)$ is a localised wave packet, $\sum_{\alpha} \psi(x)|\alpha\rangle$ is the electrons' state vector, and

$$H_x \psi|\alpha\rangle = E\psi|\alpha\rangle. \quad (18)$$

The evolution will be governed by the Schrödinger equation with the Hamiltonian of Eq. (16), and the solution for the wave function can be presented as

$$\sum_{\alpha} c_{\alpha} \Psi_{\alpha}(q, x, t)|\alpha\rangle = \psi(x) \exp\left(-i\frac{E}{\hbar}t\right) \sum_{\alpha} c_{\alpha} \phi_{\alpha}(q, t)|\alpha\rangle, \quad (19)$$

with $\phi_{\alpha}(q, t)$ being the solution to the Schrödinger equation

$$i\hbar\dot{\phi}_{\alpha} = (H_q + s_{\alpha}V_q)\phi_{\alpha}, \quad s_{\uparrow} = 1, \quad s_{\downarrow} = -1, \quad (20)$$

with the initial condition $\phi_{\alpha}(q, 0) = \phi(q)$. It is easy to show that, in view of the factorisation in Eq. (19), the guidance equation for the configuration variables Q that correspond to q is the same as what would stem from the equation (20), were the latter the genuine Schrödinger equation for a two-component spinor ϕ_{α} . Thus, in the approximation Eq. (16) considered, the evolution of Q *does not depend* on the evolution of X , the configuration variables that correspond to x . The experiment must be arranged in such a way that the wave packets $\phi_{\uparrow}(q, t)$ and $\phi_{\downarrow}(q, t)$ become nonoverlapping, and the variables Q will then enter one of them. The probabilities p_{α} of entering $\phi_{\alpha}(q, t)$, will be given by the standard expression, $p_{\alpha} = |c_{\alpha}|^2$, provided the initial distribution $p(q)$ of the variables Q is $p(q) = |\phi(q)|^2$. Remarkably, the probabilities p_{α} do not depend on the initial distribution of X .

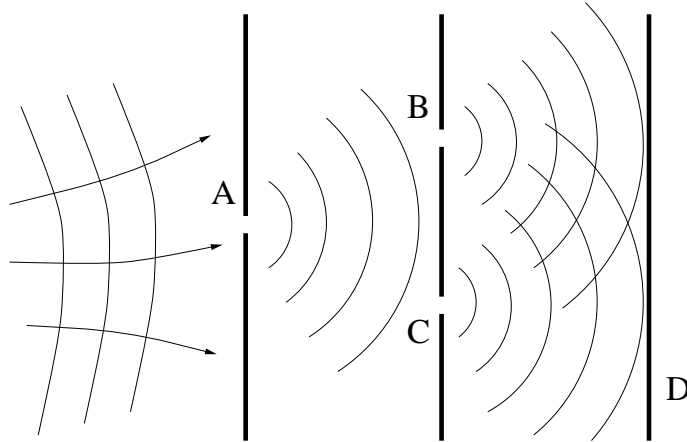


FIG. 1. Two-slit experiment

As another example consider the classical two-slit experiment (see Fig. 1). A system of collimating instruments and velocity selectors (not shown in the figure) to the left of the slit A filters out particle wave functions, letting those with wavelengths in a sufficiently narrow band to pass to reach the slit A . The role of the slit A is to produce spherical monochromatic waves in the space to the right of the slit A , this requires its dimensions to be much smaller than the wavelength of the wave function in the space to the left of the slit A . The spherical waves produced are then diffracted on a pair of slits B and C . Now, the appearance of the familiar interference pattern on the screen D will take place provided particles fall onto the slit A uniformly within the slit's range (since the wave function is also uniform on the scale of the slit dimensions, the condition $p = |\psi|^2$ will hold in the vicinity of the slit A , hence, by equivariance property it will hold also in the space to the right of A , in particular, on the screen D). But this last condition can be easily granted since the dimensions of the slit A are small. Thus, whatever of *continuous* particle distributions is realised to the left of A (its spatial scale of continuity is comparable to the spatial scale of particle wave functions), the familiar interference pattern will appear on the screen. This example suggests that the strong condition of quantum equilibrium achieved on the *universal* level may be not necessary. And, to support the agreement between the actual quantum experiments and the predictions of the pilot wave theory it is only necessary that quantum equilibrium distribution arises in *preparation* processes (natural or artificial). For example, in the case of the two-slit experiment (Fig. 1) quantum equilibrium distribution arises in the space to the right of the slit A , even though it may not take place to the left of A .

For completeness, we remark on the role of observer in quantum experiments, and in quantum theory in general. We are far from reducing a (human) observer to just a part of the universe subject to deterministic laws. Granted with a considerable freedom of action (one aspect of free will) he can influence the natural processes thus being able to make experiments. In a physical experiment the role of an observer is to put certain physical systems into contact so that they start to interact, while preventing some other systems from such a contact. In the course of a typical quantum-mechanical experiment an ensemble

of identical systems will be filtered out and then made interact with parts of the experimental equipment. Such processes usually are described in terms of external influence by an experimental equipment on an ensemble of quantum systems. This description may involve time-dependent wave functions of quantum ensembles, and then the property of equivariance will preserve the quantum equilibrium condition $p(x) = |\psi(x)|^2$ for an ensemble under consideration. In nature interaction between different more or less sharply identified systems occurs by the law of chance (from human point of view), but is described similarly to the laboratory experiments.

ACKNOWLEDGMENTS

The author acknowledges kind hospitality and highly stimulating atmosphere of IUCAA, where this paper was completed.

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